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New Surface Patches for Minimal Balance Surfaces. I. Branched Catenoids

BY WERNER FISCHER AND ELKE KOCH

Institut für Mineralogie der Universität Marburg, Hans-Meerwein-Strasse, D-3550 Marburg, Federal Republic of Germany

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Abstract

Three new families of minimal balance surfaces have been derived. For this a new kind of surface patch, *i.e.* branched catenoid, has been used. A concave polygon with one point of self-contact and a convex polygon are the two generating circuits of such a minimal balance surface.

1. Introduction

A *minimal surface* is a surface in R^3 with mean curvature zero at each of its points. A 3-periodic minimal surface without self-intersection that subdivides R^3 into two congruent regions (labyrinths) is called a *minimal balance surface* (*cf.* Fischer & Koch, 1987; Koch & Fischer, 1988). For each such minimal balance surface there exists a group-subgroup pair G–H of space groups with index 2 which uniquely describes the symmetry of that surface: G characterizes its full group of isometries whereas H consists of all those symmetry operations which do not interchange the two sides of the surface or the two labyrinths.

If s is a symmetry operation of G but not of H then all fixed points of s necessarily have to lie on each surface with symmetry G–H. To avoid self-intersection of the surface, s must not be a mirror reflection or a fourfold or sixfold rotation. As a consequence, pairs G–H with G including additional mirror planes or fourfold or sixfold rotation axes are incompatible with minimal balance surfaces.

If, on the other hand, s is a twofold rotation the corresponding entire rotation axis has to lie within each minimal balance surface with symmetry G–H. This property enables the straightforward derivation of certain kinds of minimal balance surfaces. Two kinds have already been derived completely (Fischer & Koch, 1987; Koch & Fischer, 1988): (1) all minimal balance surfaces that may be generated by skew circuits of twofold axes that are disk-like spanned (15 families); and (2) all minimal balance surfaces that may be generated by pairs of parallel flat congruent circuits of twofold axes that are catenoid-like spanned (seven families). Within the present paper three new families of minimal balance surfaces will be described that belong to a third kind.

2. Symmetry conditions for minimal balance surfaces

There exist 1156 types of group-subgroup pairs of space groups G–H with index 2 corresponding to the 1156 types of 'proper' black-white space groups. 609 of these types have been proved to be incompatible with (minimal) balance surfaces. For the other 547 types the symmetry conditions have been tabulated which must be fulfilled by each minimal balance surface with that symmetry (Koch & Fischer, 1988, Table 1). In particular, the set of twofold axes from G which do not belong to H has been identified for each pair G–H. As group-subgroup pairs of different types may define analogous such sets one has to distinguish 52 cases only. The assignment of the space-group pairs to these cases has also been done within the table mentioned above.

3. Minimal balance surfaces built up from branched catenoids

In the following, three new families of minimal balance surfaces will be described that can be derived from the sets of twofold axes belonging to cases 31 to 33 referred to in Table 1 of Koch & Fischer (1988). These sets of twofold axes disintegrate into parallel plane nets, but - in contrast to all cases discussed in previous papers - nets of two different kinds are stacked upon each other alternately. Detailed information on the three families of minimal surfaces is given in Table 1.

*Minimal balance surfaces BC*1 *made up from threefold branched catenoids*

Case 31 refers to group-subgroup pairs of type $P6_322$–$P6_3$ only. All symmetry operations from $P6_322$ that are not contained in $P6_3$ and have fixed points are twofold rotations with rotation axes parallel to the *ab* plane. The set of twofold axes defined by a certain pair $P6_322$–$P6_3$ consists of all twofold axes .2. and ..2 of $P6_322$. It disintegrates into flat parallel nets of two different kinds which are alternately arranged. The axes .2. form hexagonal nets of equilateral triangles that are oriented parallel to the coordinate axes. The rotation axes ..2 also build up hexagonal nets of equilateral triangles, but in different

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orientation. As there exist two triangles per unit cell for nets of the first kind, but six triangles per unit cell for nets of the second kind, it is impossible to construct a minimal surface from catenoid-like surface patches in the same way as has been described for cases 22 to 28 (Koch & Fischer, 1988). Instead of that, however, more complicated surface patches may be formed, called *branched catenoids*.

All catenoid-like surface patches described so far are bounded by two congruent flat convex polygons, their generating circuits (*cf.* Koch & Fischer, 1988). A branched catenoid, on the other hand, has two different generating circuits. It is bounded by a large convex polygon at one of its ends and by some smaller convex polygons with one common vertex at its other end. The smaller polygons are united to one large concave polygon with one point of self-contact.

With symmetry $P6_322$–$P6_3$ *threefold branched catenoids* may be constructed. Such a branched catenoid is bounded by one of the large triangles at one end and by three of the small triangles at the other end (Fig. 1). The small triangles meet in a common vertex which is located directly above or below the centre of the large triangle.

The minimal balance surfaces constructed from such threefold branched catenoids are designated *BC*1 (*cf.* Table 1). The generating symmetry $P6_322$–$P6_3$ of these surfaces coincides with their inherent symmetry, so that their generating linear nets are also their linear skeletal nets (*cf.* Schoen, 1970; Hyde & Andersson, 1984). According to the definition of the genus of an infinite periodic surface (*cf.* Schoen, 1970; Hyde, 1988) the genus of the *BC*1 surfaces may be calculated as 9.

Minimal balance surfaces BC2 made up from twofold branched catenoids

Case 32 refers to four types of group–subgroup pairs: $P4_2/nnm$–$P4_2nm$, $P4_222$–$P4_2$, $P4_2/nbc$–$P4_2/n$ and $P4_2/nbc$–$P4_2bc$. A corresponding set of twofold axes consists, for example, of all twofold axes of $P4_2/nnm$ parallel to the *ab* plane. Again this set disintegrates into parallel nets of two different kinds. The axes .2. form square nets oriented parallel to the coordinate axes whereas the axes ..2 give rise to square nets in diagonal orientation. As there exist two diagonal squares and four parallel squares per unit cell *twofold branched catenoids* may be constructed. They are bounded by one large diagonal square and by two small parallel squares with a common vertex (Fig. 2).

The minimal balance surfaces made up from such twofold branched catenoids are designated *BC*2. Their inherent symmetry is $P4_2/nnm$–$P4_2nm$. The generating linear nets, as described above, are also the linear skeletal nets. The genus of the *BC*2 surfaces is 7.

*BC*2 surfaces are also compatible with symmetry $P4_222$–$P4_2$ because of the group–subgroup relation with respect to their inherent symmetry. They are incompatible, however, with symmetry $P4_2/nbc$–$P4_2/n$ though analogous sets of twofold axes are defined by these pairs. The reason may be found in the roto-inversion centres $\bar{4}$ of $P4_2/nbc$ and $P4_2/n$ located in the middle of the large squares. These $\bar{4}$ centres would give rise to two branched catenoids meeting in the same large square and, as a consequence, to self-intersection of the surface within the twofold axes. In a subsequent paper a minimal surface with symmetry $P4_2/nbc$–$P4_2/n$ will be described, the surface patches of which are of another type.

Another situation occurs for $P4_2/nbc$–$P4_2bc$. The large squares of twofold axes are centred by roto-inversion centres $\bar{4}$, the small squares by $\bar{1}$. Both kinds of inversion points have to lie on all minimal surfaces with symmetry $P4_2/nbc$–$P4_2bc$ and, therefore, branched catenoids are incompatible with such space-group pairs.

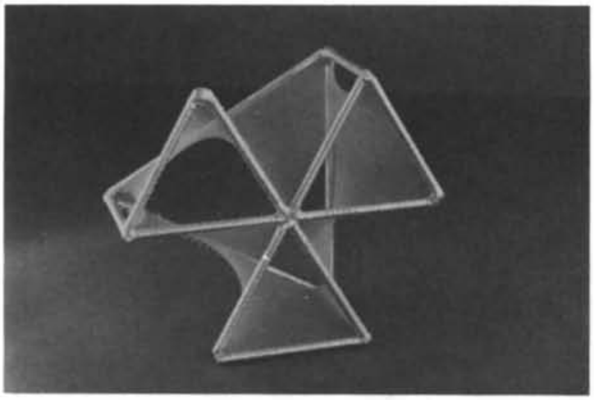

Fig. 1. Threefold branched catenoid, a surface patch of a minimal surface *BC*1 with symmetry $P6_322$–$P6_3$.

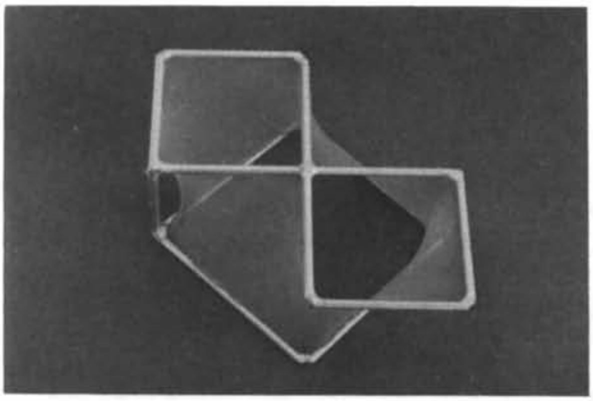

Fig. 2. Twofold branched catenoid, a surface patch of a minimal surface *BC*2 with symmetry $P4_2/nnm$–$P4_2nm$.

Minimal balance surfaces BC3 made up from fourfold branched catenoids

Case 33 refers to space-group pairs $I422$–$I4$ only. All twofold axes of $I422$ parallel to the ab plane form a corresponding set of twofold axes. Each such set disintegrates into parallel nets of two different kinds alternately arranged: triangular nets with angles 45, 45, 90° formed by all axes .2. and by half the axes ..2 and square nets in diagonal orientation made up by the other half of axes ..2. As the number of squares per unit cell is two and the number of triangles per unit cell is eight one may construct *fourfold branched catenoids*. Such a branched catenoid is bounded by a large square at one end and by four triangles sharing a common vertex (45°) at the other end (Fig. 3). The common vertex is located directly above or below the centre of the corresponding square.

The minimal balance surfaces consisting of fourfold branched catenoids are designated $BC3$. Their inherent symmetry is $I422$–$I4$ and their linear skeletal nets coincide with the generating linear nets. In spite of the relatively complicated shape of their surface patches $BC3$ surfaces have genus 6.

Common properties of BC surfaces

The existence of minimal surface patches with the shape of branched catenoids has been proved by soap-film experiments for the corresponding three families of minimal surfaces. Like a catenoid-like surface patch (*e.g.* Schoen, 1970) a branched catenoid is stable only up to a certain distance of its two generating polygons. Consequently, infinite minimal surfaces built up from branched catenoids exist only within a certain range of the axial ratio: $0 < c/a \leq c/a(\text{max.})$. The upper limits $c/a(\text{max.})$ are unknown.

As in a minimal surface built up from catenoid-like surface patches, in a minimal surface constructed from branched catenoids each convex or concave generating circuit uniquely refers to one branched catenoid. Moreover, as has been described before for catenoid-like surface patches (Koch & Fischer, 1988), the branched catenoids belonging to neighbouring polygons of the same net point in different directions.

All branched catenoids that connect the same two nets and, therefore, belong to the same layer are in parallel orientation. Branched catenoids from neighbouring layers, however, are oriented differently. For all three families of minimal surfaces described here, four layers of branched catenoids in different orientation exist within one c-translation period. Fig. 4 shows part of three layers for a $BC2$ surface. The relation between neighbouring layers is determined by the twofold axes forming the common net.

Each set of twofold axes referring to one of the cases 31, 32 or 33 forms the generating linear net for four congruent BC surfaces. Select as the first generating circuit any of the larger convex polygons formed within the set of twofold axes. Then the second concave generating circuit may belong either to the neighbouring net above or below. In both cases, there exist two further possibilities to choose the second generating circuit, so that the first circuit may be combined with a second one in four different ways giving rise to four different but congruent minimal surfaces.

Consequently, each of the BC surfaces is complementary to, *i.e.* shows the same linear skeletal net as (*cf.* Schoen, 1970; Hyde & Andersson, 1984), three other surfaces which are congruent to the first one and which show the same symmetry G–H. Therefore, all four surfaces are equivalent in the intersection group of the Euclidean normalizers $N_E(G)$ and $N_E(H)$ of G and H, respectively (*cf.* Fischer & Koch, 1983; Koch & Fischer, 1987).

This property, however, must not be generalized. Examples are known for congruent *and* complementary minimal surfaces belonging to different pairs G–H and being non-equivalent in $N_E(G) \cap N_E(H)$, therefore.

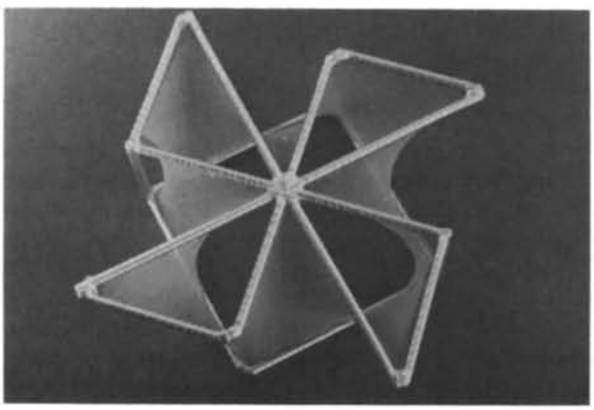

Fig. 3. Fourfold branched catenoid, a surface patch of a minimal surface $BC3$ with symmetry $I422$–$I4$.

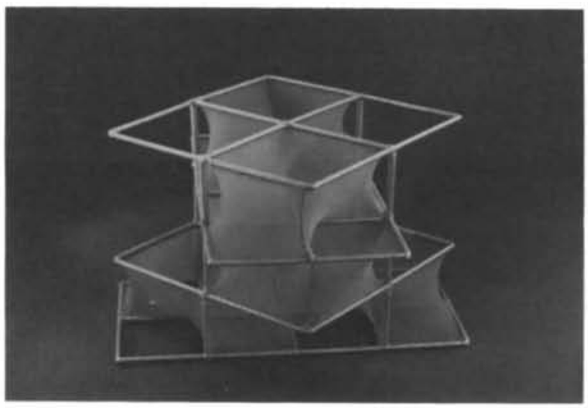

Fig. 4. A model of a minimal surface $BC2$ with symmetry $P4_2/nnm$–$P4_2nm$.

Table 1. *Minimal balance surfaces built up from branched catenoids*

Minimal balance surface	Group–subgroup pair	Genus	Surface patches: Point group	Surface patches: Generating circuits	Number of equivalent surfaces	Transformations
$BC1$	$P6_322$–$P6_3$	9	3..	3+9: $000, 100, 110/\tfrac{2}{3}\tfrac{1}{3}\tfrac{1}{4}, 00\tfrac{1}{4}, \tfrac{1}{3}\bar{\tfrac{1}{3}}\tfrac{1}{4}, \tfrac{2}{3}\tfrac{1}{3}\tfrac{1}{4}, 10\tfrac{1}{4}, \tfrac{4}{3}\tfrac{2}{3}\tfrac{1}{4}, \tfrac{2}{3}\tfrac{1}{3}\tfrac{1}{4}, 11\tfrac{1}{4}, \tfrac{1}{3}\tfrac{2}{3}\tfrac{1}{4}$	4	$m(xy0)$; $m(2x, x, z)$; $2(2x, x, z)$
$BC2$	$P4_2/nnm$–$P4_2nm$	7	$2.mm$	4+8: $\tfrac{1}{2}0\tfrac{1}{4}, 0\tfrac{1}{2}\tfrac{1}{4}, \bar{\tfrac{1}{2}}0\tfrac{1}{4}, 0\tfrac{1}{2}\tfrac{1}{4}/000, \tfrac{1}{2}00, \tfrac{1}{2}\tfrac{1}{2}0, 0\tfrac{1}{2}0, 000, \bar{\tfrac{1}{2}}00, \bar{\tfrac{1}{2}}\bar{\tfrac{1}{2}}0, 0\bar{\tfrac{1}{2}}0$	4	$m(xy\tfrac{1}{4})$; $m(0yz)$; $2(0y\tfrac{1}{4})$
$BC3$	$I422$–$I4$	6	4..	4+12: $\tfrac{1}{2}0\tfrac{1}{4}, 0\tfrac{1}{2}\tfrac{1}{4}, \bar{\tfrac{1}{2}}0\tfrac{1}{4}, 0\bar{\tfrac{1}{2}}\tfrac{1}{4}/000, \tfrac{1}{2}\tfrac{1}{2}0, 0\tfrac{1}{2}0, 000, \bar{\tfrac{1}{2}}\tfrac{1}{2}0, \bar{\tfrac{1}{2}}00, 000, \bar{\tfrac{1}{2}}\bar{\tfrac{1}{2}}0, 0\bar{\tfrac{1}{2}}0, 000, \tfrac{1}{2}\bar{\tfrac{1}{2}}0, \tfrac{1}{2}00$	4	$m(xy\tfrac{1}{4})$; $m(0yz)$; $2(0y\tfrac{1}{4})$

Information on the properties of BC surfaces is summarized in Table 1. In each case, for one of the four equivalent surfaces a pair of generating circuits is described by its vertices. Generating circuits for the other three surfaces may be calculated with the aid of the symmetry operations listed in the last column.

Minimal surfaces of the families $BC1$, $BC2$ and $BC3$ are not complementary to surfaces of other families described so far. In a subsequent paper, however, a family of minimal surfaces complementary to the $BC2$ surfaces will be presented.

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New Surface Patches for Minimal Balance Surfaces. II. Multiple Catenoids

BY ELKE KOCH AND WERNER FISCHER

Institut für Mineralogie der Universität Marburg, Hans-Meerwein-Strasse, D-3550 Marburg, Federal Republic of Germany

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Abstract

Eight new families of minimal balance surfaces are described. Their surface patches belong to a new kind, called multiple catenoids. The generating circuits of such a minimal surface are two congruent concave polygons with one point of self-contact each. The new minimal balance surfaces are complementary to other minimal balance surfaces which are built up from catenoid-like surface patches and have been known before.

1. Introduction

The symmetry of each minimal balance surface can be described by a group–subgroup pair $G \supset H$ of space groups with index 2, its inherent symmetry. The fixed points of all symmetry operations s with $s \in G$ but $s \notin H$ are necessarily contained within the surface (Fischer & Koch, 1987). Most of the minimal balance surfaces described so far have a linear skeletal net, *i.e.* a set of twofold axes defined by the corresponding space-group pair, that is embedded within the surface (*cf.* Schoen, 1970; Hyde & Andersson, 1984). Such a set of twofold axes may be used to generate a minimal balance surface (*cf.* Fischer & Koch, 1987, 1989; Koch & Fischer, 1988). Then it is called a generating linear net.

As all sets of twofold axes defined by space-group pairs with index 2 may be assigned to 52 cases (*cf.* Koch & Fischer, 1988, Table 1) at most 52 types of generating linear nets for minimal balance surfaces exist. Such a set of twofold axes may be three-dimensionally connected or not. Among the disconnected sets those ones stand out that disintegrate into parallel nets.

If all nets of such a set are congruent [cases 22 to 30 in Table 1 of Koch & Fischer (1988)] catenoid-like

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